(Elementary) Topoi

Why care about topoi

i) Place to do geometry

ii) Place to do logic

iii) Place to do set theory.

SET is a topos.

$$\begin{tikzcd} f^{-1}(B) \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$Y = \Omega = \{0,1\}$ and $B = 1$

$$\begin{tikzcd} f^{-1}(1) \arrow[r] \arrow[d] & 1 \arrow[d] \\ X \arrow[r] & \Omega \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "m"'] & 1 \arrow[d] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

INTRO

Defn: In a category $\mathcal{C}$ with a terminal object, the subobj. classifier is an obj Ω, $1 \to \Omega$ s.t. for any mono $A \hookrightarrow X$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "m"'] & 1 \arrow[d] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

Fact: $T \rightarrowtail \Omega$ s.t. $\forall\, A \xrightarrow{m} X$,

$$\begin{tikzcd} A \arrow[r, "\exists!"] \arrow[d] & T \arrow[d] \\ X \arrow[r, "\exists!"] & \Omega \end{tikzcd}$$

then, T is terminal.

$\mathrm{Mono}(\mathcal{E})$.

pullback and $\mathrm{Sub}(X)$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ {} & {} \end{tikzcd}$$

$X \xrightarrow{f} Y$

Ω^-

$\mathrm{Sub}(X) \cong \mathcal{C}(X, \Omega)$

Defn: A category $\mathcal{E}$ is said to be a topos if
i) finite limits
ii) Exponential objects X^Y $\quad (-)^Y$
iii) A subobject classifier.

Ex: Set, Set^{BG}, $\mathrm{Set}^{\mathcal{C}^{op}}$, $\mathcal{E}/X$, FinSet, 1.

Prop: i) $\mathcal{E}$ has fin. colimits.
$\Omega^- : \mathcal{E}^{op} \to \mathcal{E}$
ii) $0 \xrightarrow{\text{mono}} A$, $A \xrightarrow{\text{epi}} 0$
iii) $A, B \to X$.

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \cup B \end{tikzcd}$$

$A \cap B \cong 0$, $A \cup B \cong A + B$
iv) iso $\iff$ mono + epi
v) Any map splits into a mono + epi.

iv)
$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ A \arrow[u, "v"] \arrow[ur, "id"'] & & \end{tikzcd}$$

$cv = id$

$c(vc) = c$

$vc = id$.

$$\begin{tikzcd} B \arrow[dr, "x"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "f"'] & & \\ & A \arrow[r] \arrow[d, "m"'] \arrow[dr, "1"] & 1 \arrow[d] \\ & X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

v) $f = me$ $\quad f : A \to B$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "n"] \arrow[ddr, bend left, "s"] & \\ B \arrow[r, "y"] \arrow[drr, bend right, "t"'] & \cdot \arrow[dr, "u"] & \\ & & \cdot \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "m"] & B \arrow[r, shift left, "n"] \arrow[r, shift right, "y"'] & \cdot \\ A \arrow[u, "e"] \arrow[ur, "f"'] & & \end{tikzcd}$$

$f = hg$

$$\begin{tikzcd} A \arrow[r, "e"'] \arrow[rr, bend left, "f"] \arrow[d, equal] & M \arrow[r, "m"'] \arrow[d] & B \arrow[r, shift left, "n"] \arrow[r, shift right, "y"'] \arrow[d, equal] & \cdot \arrow[d, "u"] \\ A \arrow[r, "g"'] & N \arrow[r, "h"'] & B \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \cdot \end{tikzcd}$$

$unm = sm$
$\Vert$
$uym = tm$

m iso $\Rightarrow f$ epi

$A \xrightarrow{e'} M' \xrightarrow{m'} M \xrightarrow{m} B$

$m = m m^{-1} v$

$1 = m^{-1} v \Rightarrow m$ iso $\Rightarrow$ epi.

Generalized elements.

1. $1 \xrightarrow{x} X$

Generalized el of X is a map $S \xrightarrow{x} X$

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

$$f = g \quad \text{iff} \quad fx = gx \quad \forall x \in X$$

$\mathcal{C}$ fin. products.

$$S \underset{y}{\overset{x}{\rightrightarrows}} X \qquad m : X \times X \to X$$

$$m(x, m(y,z)) = m(m(x,y), z)$$

$$(x,y) \in X \times X \qquad x(yz) = (xy)z$$

$$(xy)^{-1} = y^{-1}x^{-1}$$

$$(xy)^{-1} = (xy)^{-1} xy y^{-1} x^{-1} = y^{-1} x^{-1}$$

$$\{ x \in X \mid fx = gx \}$$

What makes SET special? (As a topos)

i) $fx = gx \quad \forall x : 1 \to X$ for $f = g$. ((1 is a gen)

ii) $0 \not\cong 1$

well-pointed.

iii) $f : \mathbb{N} \to X \qquad x \in X \qquad f(0) = x$

$r : X \to X \qquad f(n) = r(f(n-1))$

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "x"'] & N \arrow[r] \arrow[d, "f"] & N \arrow[d, "f"] \\ & X \arrow[r, "r"] & X \end{tikzcd}$$

N is NNO

iv) AC: epis split.

<u>Thm</u>: If C is a locally-small, complete, well-pointed topos with NNO, AC, then $C \cong$ Set.