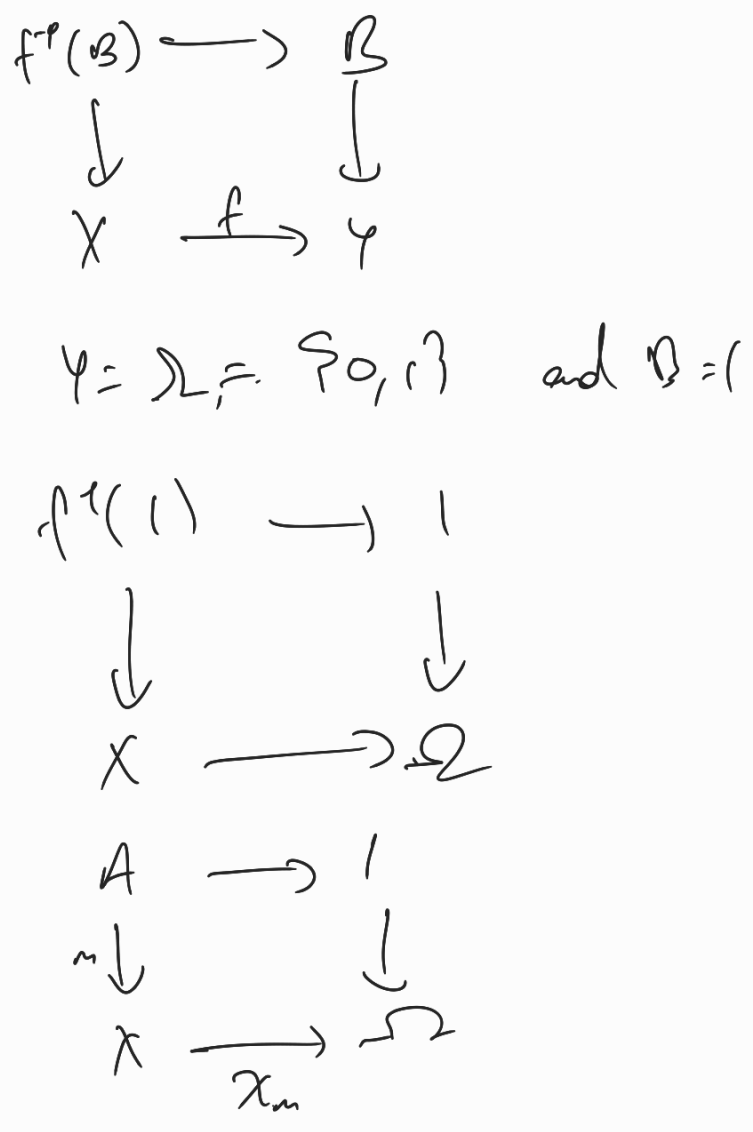


(Elementary) Topoi

Why care about topoi

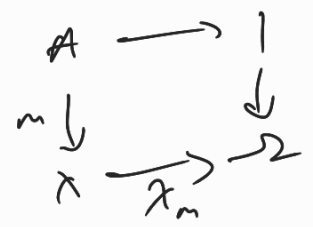
- i) Place to do geometry
- ii) Place to do logic
- iii) Place to do set theory.

SFT is a topos.



INTRO

Defⁿ: In a category \mathcal{C} with a terminal object, the subobj. classifier is an obj $\Omega, 1 \rightarrow \Omega$ s.t for any mono $A \xrightarrow{m} X$

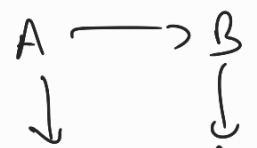


Prop: $T \rightarrow \Omega$ s.t $\forall A \xrightarrow{m} X, A \xrightarrow{\exists!} T$
 then, T is terminal.



Mono(\mathcal{E}) .

pullback and $\text{Sub}(X)$



$$\text{Sub}(X) \cong (X, \Omega)$$

$$X \xrightarrow{f} Y$$

Ω

Def. A category \mathcal{E} is said to be a topos if

- i) finite limits
- ii) Exponential objects $X \rightarrow Y \rightarrow (-)^Y$
- iii) A subobject classifier.

Ex: $\text{Set}, \text{Set}^{\text{Set}}, \text{Set}^{\text{Set}^{\text{Set}}}$
 $\mathcal{E}/X, \text{FinSet}, 1$

Prop: i) \mathcal{E} has fin. colimits.

$$\Omega : \mathcal{E}^{\text{op}} \rightarrow \mathcal{E}$$

$$\text{ii) } 0 \xrightarrow{\text{mono}} A, A \xrightarrow{\text{epi}} 0$$

$$\text{iii) } A, B \rightarrow X$$

$$A \cap B \rightarrow A$$

$$\downarrow \quad \downarrow$$

$$A \cup B \rightarrow A \cup B$$

$$A \cap B \cong 0, A \cup B \cong A + B$$

iv) iso \Leftrightarrow mono + epi

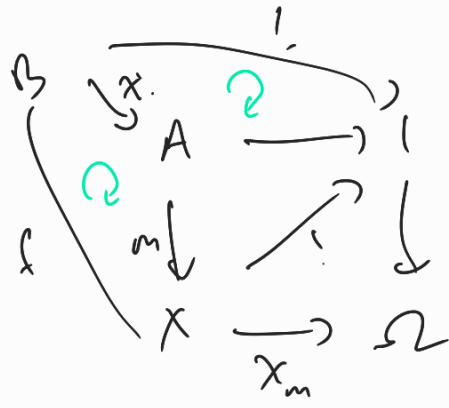
v) Any map splits into e + mono + epi.

$$\text{iv) } \begin{array}{ccc} C & \xrightarrow{f} & A \xrightarrow{g} B \\ \uparrow \text{V} & \nearrow \text{id} & \\ A & & \end{array}$$

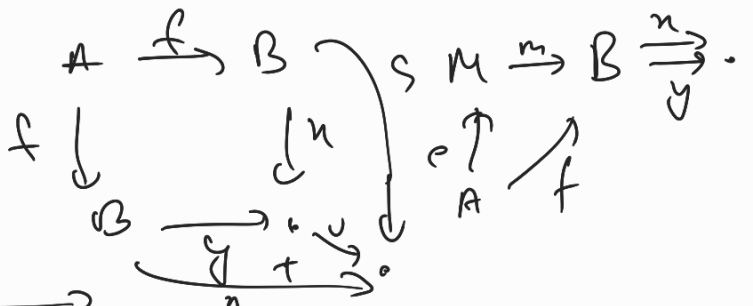
$$cu = \text{id}$$

$$c(uv) = c$$

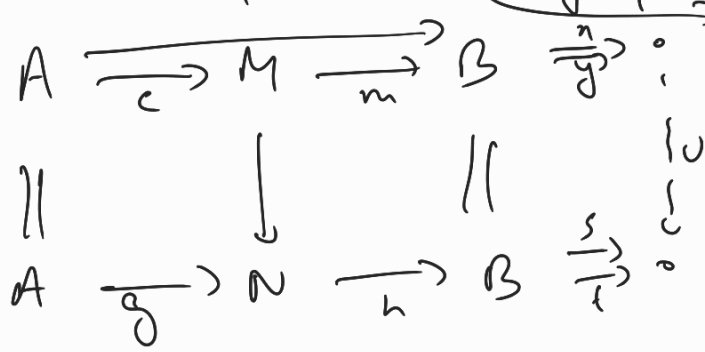
$$\Leftrightarrow vc = \text{id}$$



$$\text{v) } f = me \quad f: A \rightarrow B$$



$$f = hg$$



$$um = sm$$

$$\downarrow$$

$$ugm = tm$$

m iso $\Rightarrow f$ epi

$$A \xrightarrow{e} M' \xrightarrow{m'} M \xrightarrow{m} B$$

$$m = m \circ m \circ v$$

$$1 = m \circ v \Rightarrow m \text{ iso} \Rightarrow e \text{ fi.}$$

Generalized elements.

1. $1 \xrightarrow{x} X$

Generalize el of X is a map $S \xrightarrow{u} X$

$$X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$$

$$f = g \text{ iff } f \circ u = g \circ u \quad \forall u \in X$$

c. An. products.

$$S \xrightarrow{u} X$$

$$m: X \times X \rightarrow X$$

$$m(x, m(y, z)) = m(m(x, y), z)$$

$$(x, y) \in X \times X$$

$$x(y, z) = (xy)z$$

$$(xy)^t = y^t x^t$$

$$(xy)^t = (xy)^t x y^t x^t = y^t x^t$$

$$\{u \in X \mid f \circ u = g \circ u\}$$

What makes SET special? (As a topos)

i) $f \circ u = g \circ u \quad \forall u: 1 \rightarrow X$ for $f = g$. (i.e. a gen)

ii) $0 \neq 1$

well-pointed.

iii) $f: \mathbb{N} \rightarrow X$

$$x \in X$$
$$v: X \rightarrow X$$

$$f(0) = x$$
$$f(n) = v(f(n-1))$$

$$\begin{array}{ccc}
 1 \circlearrowright & N & \rightarrow N \\
 & \downarrow f & \downarrow f \\
 \alpha & X & \xrightarrow{r} X
 \end{array}$$

N is NNO

(iv) AC: epis split.

Thm: If C is a locally-small, complete, well-pointed topos with NNO , AC, then $C \cong \text{Set}$.